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BPS-type Equations in the Non-anticommutative N=2 Supersymmetric U(1) Gauge Theory [1]

Sergei V. Ketov [2] and Shin Sasaki [3]

Department of Physics, Faculty of Science
Tokyo Metropolitan University
1–1 Minami-osawa, Hachioji-shi
Tokyo 192–0397, Japan

Abstract

We investigate the equations of motion in the four-dimensional non-anticommutative N=2 supersymmetric $U(1)$ gauge field theory, in the search for BPS configurations. The BPS-like equations, generalizing the abelian (anti)self-duality conditions, are proposed. We prove full solvability of our BPS-like equations, as well their consistency with the equations of motion. Certain restrictions on the allowed scalar field values are also found. Surviving supersymmetry is briefly discussed too.

[1]Supported in part by the JSPS and the Volkswagen Stiftung

[2]Email address: ketov@phys.metro-u.ac.jp
[3]Email address: shin-s@phys.metro-u.ac.jp

1 Introduction

Supersymmetric field theories in *Non-AntiCommutative* (NAC) superspace stretch limits of the conventional supersymmetric field theories formulated in the standard superspace with anticommutative (Grassmann) spinor coordinates. Unlike the usual (spacetime) non-commutative field theories, where solving non-commutative field equations represents a formidable task [1, 2], the NAC deformations can be essentially nilpotent so that they may lead to solvable (though non-trivial) field equations of motion.

To the best of our knowledge, possible non-commutativity of spacetime (bosonic) coordinates x^m,

$$[x^m, x^n]_\star = i\theta^{mn} , \tag{1.1}$$

was first proposed (in a published paper) in 1947 [3], though Heisenberg was known to be privately suggesting this idea in the 30's. Non-anticommutativity of the fermionic (Grassmann) superspace coordinates θ^α (also known as the *quantum superspace*),

$$\{\theta^\alpha, \theta^\beta\}_\star = C^{\alpha\beta} , \tag{1.2}$$

was first proposed in 1981 [4] (see also ref. [5] for more recent developments). The non-commutativity (1.1) is well known to appear in superstring theory, in the presence of a constant background of the NS-NS antisymmetric B-field [6]. Most recent interest to the NAC supersymmetric field theories is due to their relevance in describing some superstring effective actions in certain supergravity backgrounds [7, 8].

The *nilpotent* quantum superspace arises when merely a chiral part of the fermionic superspace coordinates becomes NAC, whereas bosonic superspace coordinates still commute (in some basis) [8]. This is only possible when the anti-chiral fermionic coordinates ($\bar{\theta}$) are not complex conjugates to the chiral ones, $\bar{\theta} \neq (\theta)^*$, which is the case in Euclidean or Atiyah-Ward spacetimes with the signature $(4, 0)$ and $(2, 2)$, respectively. The Euclidean signature is relevant to instantons and superstrings [8], whereas the Atiyah-Ward signature is relevant to the critical N=2 string models [9, 10] and supersymmetric self-dual gauge field theories in 2+2 dimensions [11]. In the case of N=1 supersymmetric gauge field theories in the nilpotent NAC superspace subject to eq. (1.2), $(\frac{1}{2}, \frac{1}{2})$ supersymmetry is always broken by $C^{\alpha\beta} \neq 0$ to $(\frac{1}{2}, 0)$ supersymmetry, while the change of the Lagrangian is polynomial in $C^{\alpha\beta}$ [8].

Extended supersymmetry is expected to bring more constraints to the NAC supersymmetric field theories. The NAC extension of the $N = (1, 1)$ supersymmetric gauge field theory along the lines of ref. [8] was constructed in ref. [12], with the

deformed action having merely $(\frac{1}{2}, 0)$ supersymmetry. There is another problem with the equations of motion in that theory. Let's consider the NAC extension of the N=2 supersymmetric U(1) gauge theory having the Lagrangian [12]

$$
\begin{aligned}
L \;=\; & -\frac{1}{4}F^{mn}F_{mn} - i\bar{\lambda}\bar{\sigma}^m\partial_m\lambda + \frac{1}{2}D^2 - \frac{i}{2}C^{mn}F_{mn}\bar{\lambda}\bar{\lambda} \\
& - \partial^m\bar{A}\partial_m A - i\bar{\psi}\bar{\sigma}^m\partial_m\psi + \bar{F}F + iC^{mn}F_{mn}\bar{A}F \; .
\end{aligned}
\tag{1.3}
$$

Being compared to the N=1 case represented by the first line of eq. (1.3), the terms in the second line are needed for N=2 extension. However, the equation of motion for $\bar{F}$ implies $F = 0$ which, in its turn, gives rise to decoupling of $(A, \bar{A}, \psi, \bar{\psi})$. The only nontrivial deformation is then exactly the same as in the N=1 case [8].

When one wants a nontrivial NAC extension of the N=2 supersymmetric gauge field theory, one thus has to consider other deformations. The most general nilpotent deformation of $N = (1, 1) = 2 \times (\frac{1}{2}, \frac{1}{2})$ supersymmetry is given by [13, 14]

$$
\{\theta^\alpha_i, \theta^\beta_j\}_\star = \delta^{(\alpha\beta)}_{(ij)}C^{(\alpha\beta)} + 2iP\varepsilon^{\alpha\beta}\varepsilon_{ij} \quad \text{(no sum!)} \; , \tag{1.4}
$$

where $\alpha, \beta = 1, 2$ are chiral spinor indices, $i, j = 1, 2$ are the indices of the internal R-symmetry group $SU(2)$, while $C^{\alpha\beta}$ and P are some constants. Our strategy is to keep N=2 supersymmetry at the level of the NAC Lagrangian, and then search for its BPS-type solutions that are supposed to break some part of supersymmetry, as is usual in field and string theories. As was noticed in ref. [14], N=2 supersymmetry gives us the unique opportunity of a nilpotent NAC deformation preserving both N=2 supersymmetry, R-symmetry and Lorentz invariance, when keeping only $P \neq 0$ while setting $C^{\alpha\beta} = 0$ in eq. (1.4). We believe that the NAC deformation parameter P may be related to the vacuum expectation value of some RR-type scalar in the superstring compactification, though we are not going to pursue the superstring connection in this paper. Instead, we consider the equations of motion in the NAC-deformed N=2 supersymmetric U(1) gauge theory, and find its BPS-type equations, in order to demonstrate some advantages of the P-deformation versus the C-deformation in the case of extended supersymmetry. The non-abelian gauge groups in the present context will be considered elsewhere [15]. Similar BPS-type equations in the N=1/2 supersymmetric gauge theories with a non-singlet NAC deformation $(C_{\alpha\beta} \neq 0)$ were derived in ref. [16].

Our paper is organized as follows. Sect. 2 is a brief review of the main results of ref. [14] which is the pre-requisite to our paper. This section also serves as a technical introduction. In sect. 3 we establish the BPS-type equations in our theory and demonstrate their consistency with the equations of motion. In sect. 4 we discuss a general solution to our equations and their symmetries. Sect. 5 is our conclusion.

2 Lagrangian

For definiteness, we work in flat Euclidean spacetime, though we use the notation and conventions common to N=2 superspace with Minkowski spacetime signature (see ref. [17] for details about our notation).

Our NAC N=2 superspace with the coordinates $(x^m, \theta^i_\alpha, \bar{\theta}^{\dot{\alpha}}_i)$ is defined by eq. (1.4), with $C^{\alpha\beta} = 0$ and $P \neq 0$, as the *only fundamental* non-trivial (anti)commutator amongst the N=2 superspace coordinates. This choice preserves the so-called G-analyticity that is a fundamental feature of N=2 supersymmetry [14], while keeping the bosonic spacetime coordinates (in the G-analytic basis) to be commuting.

The unique star product in the NAC N=2 superspace (1.4), which preserves N=2 supersymetry, R-symmetry and 'Lorentz' invariance, is given by [14]

$$A \star B = A \exp\left(iP\varepsilon^{\alpha\beta}\varepsilon^{ij} \overleftarrow{D}_{i\alpha} \vec{D}_{j\beta} \right) B \quad , \tag{2.1}$$

where $D_{i\alpha}$ are the standard N=2 chiral supercovariant derivatives. The star product (2.1) allows us to introduce N=2 *anti-chiral* superfields (defined by $D_{i\alpha}\bar{\Phi} = 0$). In the N=2 superspace *anti-chiral* basis, the N=2 chiral supercovaraint derivatives are simply given by $D_{i\alpha} = -\partial/\partial\theta^{i\alpha}$, while the standard abelian N=2 anti-chiral superfield strength in components (to be defined this way after expanding in powers of $\bar{\theta}$) reads

$$\begin{aligned} \overline{W}(x_R, \bar{\theta}) = \quad & \bar{\phi} + \bar{\theta}_{i\dot{\alpha}} \bar{\lambda}^{i\dot{\alpha}} + \frac{1}{8\sqrt{2}} (\bar{\theta}_i \tilde{\sigma}^{mn} \bar{\theta}^i) F^-_{mn} \\ & + \frac{1}{2\sqrt{2}} \bar{\theta}_{ij} D^{ij} + (\bar{\theta}^3)^i_{\dot{\alpha}} (\tilde{\sigma}^m)^{\dot{\alpha}\alpha} \partial_m \lambda_{i\alpha} - (\bar{\theta})^4 \Box \phi \quad . \end{aligned} \tag{2.2}$$

The scalars $\bar{\phi}$ and ϕ, as well as the chiral spinors $\bar{\lambda}^i_{\dot{\alpha}}$ and $\lambda_{i\alpha}$, are the *independent* fields in Euclidean or Atiyah-Ward spacetimes, i.e. they are not related by complex conjugation, contrary to Minkowski spacetime. The self-dual and anti-selfdual parts of the Maxwell field strength, F^+_{mn} and F^-_{mn} respectively, are defined by $F^\pm = \frac{1}{2}(F \pm {}^*F)$, where ${}^*F$ is the dual field strength, ${}^*F^{mn} = \frac{1}{2}\varepsilon^{mnpq}F_{pq}$. The spacetime derivatives in eq. (2.2) at the $(\bar{\theta}^3)$ and $(\bar{\theta})^4$ terms are needed to provide canonical dimensions to the component fields λ and ϕ, while they are also consistent with the Bianchi identities on F_{mn} and N=2 supersymmetry. The $SU(2)$ triplet D_{ij} are the auxiliary fields.

The free action of the N=2 supersymmetric $U(1)$ gauge theory in the standard (undeformed) N=2 superspace is given by [18]:

$$S_{\text{free}} = \tfrac{1}{2} \int d^4x_R d^4\bar{\theta}\, \overline{W}^{\,2} \quad . \tag{2.3}$$

The most general deformation of the action (2.3), compatible with N=2 supersymmetry, R-symmetry and 'Lorentz invariance', and having no higher derivatives, is parameterized by an arbitrary real function $f(\overline{W})$,

$$S_f = \tfrac{1}{2}\int d^4x_R d^4\bar{\theta}\; f(\overline{W}) \equiv \int d^4x L_f \quad . \tag{2.4}$$

It is straightforward to calculate the component Lagrangian associated with the action (2.4). The result in our notation is given by

$$\begin{aligned} L_f \;=\; & -\tfrac{1}{2}f'(\bar{\phi})\Box\phi - \tfrac{1}{4}f''(\bar{\phi})(F^-_{mn})^2 - \tfrac{1}{2}f''(\bar{\phi})\bar{\lambda}_i\tilde{\sigma}^m i\partial_m\lambda^i + \tfrac{1}{8}f''(\bar{\phi})D^{ij}D_{ij} \\ & - \frac{1}{4\sqrt{2}}f'''(\bar{\phi})\bar{\lambda}_{i\dot{\alpha}}\bar{\lambda}_{j\dot{\beta}}\left[\tfrac{1}{4}\varepsilon^{ij}(\tilde{\sigma}^{mn})^{\dot{\alpha}\dot{\beta}}F^-_{mn} + \varepsilon^{\dot{\alpha}\dot{\beta}}D^{ij}\right] + \tfrac{1}{2}f''''(\bar{\phi})(\bar{\lambda})^4 \ , \end{aligned} \tag{2.5}$$

where the primes denote differentiations with respect to $\bar{\phi}$.

Since the NAC P-deformation does not break N=2 supersymmetry, R-symmetry and 'Lorentz invariance' either, the resulting non-linear action in the standard N=2 superspace must belong to the family (2.4). An actual calculation of the effective function f requires the explicit use of the N=2 gauge superfield pre-potentials in the non-abelian setup, which is only possible in N=2 harmonic superspace. This calculation was done in ref. [14] with the following result:

$$f(\overline{W}) = \frac{\overline{W}^{\,2}}{(1+P\overline{W})^2} \ , \quad \text{or, effectively,} \quad \overline{W}_{\rm nac} = \frac{\overline{W}}{(1+P\overline{W})} \quad . \tag{2.6}$$

This is to be compared to the exact solution to the Seiberg-Witten map [6] in the abelian (but spacetime non-commutative) case [19]:

$$(F_{\rm nc})_{mn} = \frac{F_{mn}}{1+F\cdot\theta} \ , \quad \text{where} \quad F\cdot\theta = \theta^{mn}F_{mn} \quad . \tag{2.7}$$

Both effective functions in eqs. (2.6) and (2.7) are apparently the same, though eq. (2.6) is Lorentz-invariant and N=2 supersymmetric whereas eq. (2.7) is not.

3 BPS equations

It is straightforward to calculate the Euler-Lagrange equations of motion from the Lagrangian (2.5). The variations with respect to ϕ and λ^i give rise to the equations, respectively,

$$\Box(f'(\bar{\phi})) = 0 \quad \text{and} \quad i\sigma^m\partial_m\left(f''(\bar{\phi})\bar{\lambda}_i\right) = 0 \quad . \tag{3.1}$$

The variation with respect to the abelian gauge field A^n yields

$$\partial^m \left[f''(\bar\phi) F^-_{mn} + \frac{1}{8\sqrt{2}} f'''(\bar\phi)(\bar\lambda^i \tilde\sigma_{mn} \bar\lambda_i) \right] = 0 \ . \tag{3.2}$$

The variation with respect to D_{ij} gives rise to the algebraic constraint

$$f''(\bar\phi) D^{ij} - \frac{1}{\sqrt{2}} f'''(\bar\phi)(\bar\lambda^i \bar\lambda^j) = 0 \quad . \tag{3.3}$$

The variation with respect to $\bar\lambda_{i\dot\alpha}$ yields the fermionic equation of motion

$$f''(\bar\phi)(i\tilde\sigma^m \partial_m \lambda^i)_{\dot\alpha} + \frac{1}{\sqrt{2}} f'''(\bar\phi)\bar\lambda_{\dot\alpha j} D^{ij} - f''''(\bar\phi)(\bar\lambda^3)^i_{\dot\alpha} = 0 \ , \tag{3.4}$$

while the variation with respect to $\bar\phi$ leads to the bosonic equation of motion

$$\begin{aligned} \Box\phi \ = \ & -\frac{f'''(\bar\phi)}{2f''(\bar\phi)} \Big[(F^-_{mn})^2 + 2i(\bar\lambda_i \sigma^m \partial_m \lambda^i) - \frac{1}{2} D^{ij} D_{ij} \\ & + \frac{f''''(\bar\phi)}{\sqrt{2} f'''(\bar\phi)} \left\{ \tfrac{1}{4}(\bar\lambda^i \tilde\sigma^{mn} \bar\lambda_i) F^-_{mn} + (\bar\lambda_i \bar\lambda_j) D^{ij} \right\} - \frac{2f'''''(\bar\phi)}{f'''(\bar\phi)} (\bar\lambda)^4 \Big] \ . \end{aligned} \tag{3.5}$$

Equations (3.1) amount to the *free* Klein-Gordon and Dirac equations of motion,

$$\Box\bar\phi_{\rm new} = 0 \quad \text{and} \quad i\sigma^m \partial_m \bar\lambda^i_{\rm new} = 0 \ , \tag{3.6}$$

after the field redefinitions

$$f'(\bar\phi) = \bar\phi_{\rm new} \quad \text{and} \quad f''(\bar\phi)\bar\lambda^{\dot\alpha i} = \bar\lambda^{\dot\alpha i}_{\rm new} \quad . \tag{3.7}$$

Equations (3.3) clearly determine the auxiliary fields as follows:

$$D^{ij} = -\frac{f'''(\bar\phi)}{\sqrt{2} f''(\bar\phi)} (\bar\lambda^i \bar\lambda^j) \quad . \tag{3.8}$$

The equations of motion (3.2), (3.4) and (3.5) are non-trivial, whose interaction terms are entirely fixed by the NAC deformation $P \neq 0$. It is worth mentioning that the new NAC-generated contributions (i.e. those vanishing at $P = 0$) correspond to the terms above that contain the third or higher derivatives of f.

The Euler-Lagrange equations of motion above have the second-order derivatives for bosons and the first-order derivatives for fermions. The BPS equations should be of the first order in the bosonic fields, they are supposed to be some deformed version of the abelian selfduality or anti-selfduality equations, they should also be derivable by minimizing the Euclidean action, while they are to be consistent with the Euler-Lagrange equations of motion too. Since the NAC deformation is essentially chiral, the selfdual and anti-selfdual cases differ, so that they are to be considered separately.

NAC *selfduality* equations. The Lagrangian (2.5) does not depend upon F^+_{mn} at all, so that the undeformed self-duality condition is the only possibility,

$$F^+_{mn} = 0 \ . \tag{3.9}$$

A consistency between eqs. (3.2) and (3.9) is only possible when $\bar{\phi} = const.$ because the required condition $\partial_m(\bar{\lambda}_i\tilde{\sigma}^{mn}\bar{\lambda}_i) = 0$ then appears to be the consequence of the free Dirac equation of motion in eq. (3.1). The spinor field $\bar{\lambda}_{i\dot{\alpha}}$ represents the undeformed anti-chiral fermionic zero modes in this case.

NAC *anti-selfduality* equations. Arranging the perfect square involving F^-_{mn} in the Lagrangian (2.5) and minimizing the action give rise to the P-deformed anti-selfduality equation

$$f''(\bar{\phi})F^-_{mn} + \frac{1}{8\sqrt{2}}f'''(\bar{\phi})(\bar{\lambda}^i\tilde{\sigma}_{mn}\bar{\lambda}_i) = 0 \ . \tag{3.10}$$

This is obviously consistent with the equation of motion (3.2). Substituting eqs. (3.3) and (3.10) back into the Lagrangian (2.5) eliminates the terms quadratic in $\bar{\lambda}$, so that the remaining fermionic interaction terms become quartic in $\bar{\lambda}$,

$$L_{\text{fermi-int.}} = F(\bar{\phi})(\bar{\lambda})^4 \ , \quad \text{where} \quad F(\bar{\phi}) = -\frac{3f'''(\bar{\phi})^2}{2f''(\bar{\phi})} + \frac{1}{2}f''''(\bar{\phi}) \ . \tag{3.11}$$

Similarly, the rest of equations (3.4) and (3.5) is now given by

$$i\tilde{\sigma}^m\partial_m\lambda^i - \frac{2F(\bar{\phi})}{f''(\bar{\phi})}(\bar{\lambda}^3)^i = 0 \tag{3.12}$$

and

$$\Box\phi = J(\bar{\phi})(\bar{\lambda})^4 \ , \quad \text{with} \quad J(\bar{\phi}) = \frac{4f'''(\bar{\phi})}{f''(\bar{\phi})^2}F(\bar{\phi}) - \frac{2}{f'''(\bar{\phi})}\frac{\partial F(\bar{\phi})}{\partial\bar{\phi}} \ . \tag{3.13}$$

After taking into account eq. (2.6) we find

$$F(\bar{\phi}) = \frac{12P^2(5P^2\bar{\phi}^2 - 10P\bar{\phi} + 6)}{(1 + P\bar{\phi})^6(-1 + 2P\bar{\phi})} \tag{3.14}$$

and

$$J(\bar{\phi}) = \frac{24P^3(5P^3\bar{\phi}^3 - 15P^2\bar{\phi}^2 + 24P\bar{\phi} - 19)}{(1 + P\bar{\phi})^3(-1 + P\bar{\phi})^3} \ . \tag{3.15}$$

The function (3.14) becomes singular at $P\bar{\phi} = -1$ and $P\bar{\phi} = +\frac{1}{2}$, where our effective description of NAC breaks down. [4] Hence, we get limits of the allowed real (physical) values of the field $\bar{\phi}$,

$$-1 < P\bar{\phi} < \frac{1}{2} \ . \tag{3.16}$$

[4] We assume, for definiteness, that $P \geq 0$.

The quadratic polynomial $5P^2\bar{\phi}^2 - 10P\bar{\phi} + 6$ in the numerator of the function F in eq. (3.14) is always positive for all real values of $\bar{\phi}$, so that the fermionic equation (3.12) cannot become a free Dirac equation unless $P = 0$ or $P \to \infty$. The graphs of the functions $F(\bar{\phi})$ and $J(\bar{\phi})$ are given in Fig. 1.

In the anti-commutative limit, $P \to +0$, the obstructions (3.16) disappear, as they should.

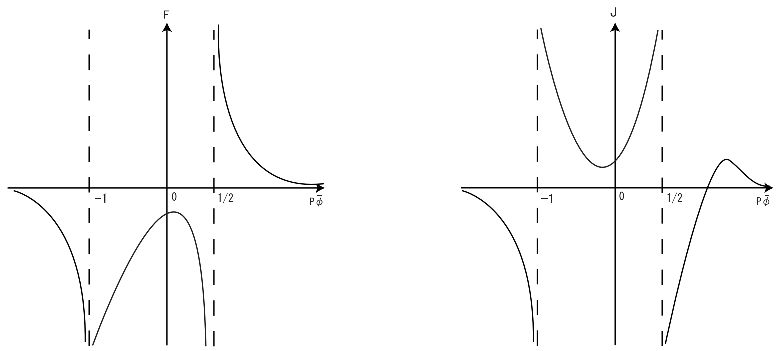

Figure 1: Graphs of the functions F in eq. (3.14) and the function J in eq. (3.15).

4 Solutions and Symmetries

The remarkable fact about our BPS-like equations (sect. 3) is their solvability. A general solution to the field equations can be written down explicitly, despite of their apparently non-linear form. This seems to be the very special feature of our abelian NAC theory that does not seem to have an analogue in Minkowski spacetime.

Let's begin with equations (3.1). They can be brought to the form (3.6) after the algebraic field redefinition (3.7). The most general solutions to the *free* Klein-Gordon and Dirac equations (3.6) are well known, either in Euclidean or Atiyah-Ward spacetime. Inverting the algebraic relations (3.7) we can get both functions $\bar{\phi}$ and $\bar{\lambda}$ in terms of the general solution to eq. (3.6). Further, eq. (3.10) actually delivers us a solution to F^-_{mn} in terms of the already known functions $\bar{\phi}$ and $\bar{\lambda}$. Now the remaining equations (3.12) and (3.13) take the form of the Dirac and Klein-Gordon equations on λ and ϕ, respectively, with the *known* sources. Hence, they can also be easily solved explicitly, by using the standard Green functions of the *free* Klein-Gordon and Dirac operators.

The N=2 supersymmetry transformations of all the field components in the N=2 superfield (2.5) follow by direct calculation from N=2 superspace (see e.g., ref. [17]). We find

$$\begin{aligned} \delta\bar{\phi} &= -\bar{\varepsilon}_{\dot{\alpha}j}\bar{\lambda}^{\dot{\alpha}j} \ , \\ \delta\bar{\lambda}_{\dot{\alpha}i} &= \frac{1}{4\sqrt{2}}(\tilde{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}\bar{\varepsilon}_i^{\dot{\beta}}F^-_{mn} + \frac{1}{\sqrt{2}}\bar{\varepsilon}^j_{\dot{\alpha}}D_{ij} - 2i\varepsilon^{\alpha}_i(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\phi} \ , \end{aligned} \tag{4.1}$$

and similarly (by formal 'complex conjugation') for ϕ and λ, as well as

$$\delta F^-_{mn} = \frac{-i}{\sqrt{2}}\left\{(\varepsilon_j\sigma_{[m}\partial_{n]}\bar{\lambda}^j) - (\bar{\varepsilon}_j\tilde{\sigma}_{[m}\partial_{n]}\lambda^j) - \tfrac{1}{2}\varepsilon_{mnpq}\left[(\varepsilon_j\sigma_p\partial_q\bar{\lambda}^j) - (\bar{\varepsilon}_j\tilde{\sigma}_p\partial_q\lambda^j)\right]\right\} \ , \tag{4.2}$$

where we have introduced the infinitesimal anticommuting (Grassmann) spinor parameters $(\varepsilon_{i\alpha}, \bar{\varepsilon}^{i\dot{\alpha}})$ of rigid N=2 supersymmetry.

It is easy to see that the self-duality condition (3.9) implies $\lambda_{i\alpha} = 0$ by supersymmetry, as well as $\bar{\phi} =$ const., as it should have been expected from supersymmetry. Hence, some supersymmetry may be preserved in this case.

As regards the anti-selfdual case, requiring a half of the N=2 supersymmetry transformations in eq. (3.10) to vanish leads to one of following fermionic equations:

$$\left(\tilde{\sigma}_{[m}\partial_{n]}\lambda^i\right)^- = \frac{if'''(\bar{\phi})}{8f''(\bar{\phi})}\bar{\lambda}^{ij}\left(\tilde{\sigma}_{mn}\bar{\lambda}_j\right) \ , \tag{4.3a}$$

or

$$\left(\sigma_{[m}\partial_{n]}\bar{\lambda}_i\right)^- = -\frac{2f'''(\bar{\phi})}{f''(\bar{\phi})}\left(\sigma_{[m}\partial_{n]}\bar{\phi}\right)^-\bar{\lambda}_i \ , \tag{4.3b}$$

where we have used the fact that the rigid N=2 supersymmetry parameters $\varepsilon_{i\alpha}$ and $\bar{\varepsilon}^{i\dot{\alpha}}$ are independent, as well as the notation

$$X^-_{mn} = X_{mn} - \tfrac{1}{2}\varepsilon_{mnpq}X_{pq} \tag{4.4}$$

to denote the anti-selfdual part of an antisymmetric tensor X_{mn}. The fermionic BPS-like conditions (4.3) are complementary to the bosonic BPS-like equation (3.10), as long as some part of supersymmetry is preserved.

The simplest solution to eq. (4.3b) is given by $\bar{\lambda}^j_{\dot{\alpha}} = 0$. This is consistent with the equations of motion, it implies $\bar{\phi} =$ const. and $F^-_{mn} = 0$, while it preserves half of supersymmetry. Unfortunately, this solution does not lead to a non-trivial deformation of the abelian anti-selfduality equation [11]. It remains to be seen whether there are other non-trivial solutions to eq. (4.3) that would be consistent with the equations of motion and, hence, preserve some part of supersymmetry by construction.

5 Conclusion

Our considerations in this paper were entirely classical. It would be interesting to investigate the role of quantum corrections both in quantum field theory and in string theory (e.g. by using the geometrical engineering). In particular, the standard Seiberg-Witten solutions [20] in the Coulomb branch of quantum N=2 super-Yang-Mills theories formally belong to the class (2.4) considered in this paper, so that it is conceivable to conjecture that quantum corrections to the NAC solution (2.6) may be linked to the Seiberg-Witten theory.

Upon dimensional reduction to *two* spacetime dimensions, the field theory (2.5) with *any* function f gives rise to the two-dimensional N=4 supersymmetric non-linear sigma-model with a non-trivial torsion (or a generalized Wess-Zumino term), whose geometry and renormalization were investigated in ref. [21]. In particular, as was shown in ref. [21], all those two-dimensional supersymmetruic non-linear sigma-models are always ultra-violet *finite*, as the quantum field theories, in all loops.

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